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SEVERAL FINDINGS ON FRACTIONAL (p, q)-DERIVATIVES FOR THE (p, q)-VARIANT OF PRATHIMA'S MULTIVARIABLE I-FUNCTION

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Abstract: This article gives the (p, q)-analogue of the modified multivariable Prathima's I-function and explore its properties under the (p, q)-analogue derivative fractional operator. Additionally, we discuss various corollaries considering the (p, q) -analogue counterparts of various multivariable H-functions and I-functions, both in one and two variables.

Keywords and Phrases: Multiple Mellin-Barnes contour integrals, (p, q)-analogue of multivariable I-function, (p, q) -analogue of multivariable H-function, (p, q) -analogue I-function of two variables, (p, q)-analogue H-function of two variables, (p, q)-analogue derivative fractional operator.

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1. Introduction and Preliminaries

By including double number of parameters in the (p, q)-identities we have various choices for manipulations. It was observed that several q-results can be generalized directly to (p, q)-results. Moreover once we have the (p, q)-results, the q-results can be obtained more easily by mere substitutions for the parameters instead of any limiting process as required in the usual q-theory [24]. Recently, Ahmad [2] has introduced and studied the (p, q)-analogue of I-function introduced by [32] (see also, [12, 13, 31]). In this paper, we give two results about the (p, q)-analogue of multivariable Prathima's I-function.

Sadjang [30] introduced the shifted factorial as follows:

$$(x \ominus a)_{p,q}^{n} = (x-a)(px-aq)\left(p^{2}x-aq^{2}\right)\cdots\left(xp^{n-1}-aq^{n-1}\right), \tag{1.1}$$

and

$$(x \oplus a)_{p,q}^{n} = (x+a)(px+aq)\left(p^{2}x+aq^{2}\right)\cdots\left(xp^{n-1}+aq^{n-1}\right). \tag{1.2}$$

We can extend above formulas in the following manner:

$$(x \ominus a)_{p,q}^{n} = \prod_{k=0}^{\infty}\left(xp^{k}-aq^{k}\right), \tag{1.3}$$

and

$$(x \oplus a)_{p,q}^{n} = \prod_{k=0}^{\infty}\left(xp^{k}+aq^{k}\right). \tag{1.4}$$

For $x \in \mathbb{C}$, the (p, q)-Gamma function [29] is given by

$$\Gamma_{p,q}(x) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^{x} \ominus q^{x})_{p,q}^{\infty}}(p-q)^{x} \quad (0<q<p). \tag{1.5}$$

Taking $p = 1$, then $\Gamma_{p,q}$ reduces to Γ_q.
The (p, q)-Gamma function fulfils the following functional equation:

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\,\Gamma_{p,q}(x). \tag{1.6}$$

If $n \geqslant 0$ (n is an integer), we obtain the relation

$$\Gamma_{p,q}(n+1) = [n]_{p,q}!. \tag{1.7}$$

Arik et al. [5] also defined the (p, q) -Beta function, given as

$$B_{p,q}(x,y) = \frac{\Gamma_{p,q}(x)\,\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)}. \tag{1.8}$$

The (p,q)-derivative of the function $f(x)$ is defined as follows [3]:

$$D_{p,q}f\left(x\right)=\frac{f\left(px\right)-f\left(qx\right)}{\left(p-q\right)x}\text{ for }x\neq 0,\tag{1.9}$$

where $D_{p,q}f\left(0\right)=f'\left(0\right)$, if $f\left(x\right)$ is differentiable at $x=0$. We have the following result:

Lemma 1.1. [2, eqn.(7.1.8), ch.7, p.104]

$$D_{p,q}^{n}\left(x^{\mu}\right)=\frac{\Gamma_{p,q}\left(\mu+1\right)}{\Gamma_{p,q}\left(\mu-n+1\right)}x^{\mu-n}\quad\left(\Re\left(\mu\right)+1>0\right).\tag{1.10}$$

Now, we define the (p,q)-analogue of the modified multivariable I-function defined by Prathima et al. [26].

Remark 1.1. *Basic analogue of several special functions has been defined and studied by many authors, for e.g.* [15, 16, 17, 18, 19, 31].

2. (p,q)-analogue of modified multivariable I-function

Recently, I-function of several variables has been introduced and studied by Prathima et al. [26], it's an extension of the H-function of several variables defined by [33, 34]. In this paper, we define a new function, the (p,q)-analogue of multivariable I-function. It is defined by (p,q)-Mellin-Barnes multiple contour integrals, given as

$$\mathbb{I}\left(z_1,\cdots,z_r;p,q\right)=I^{0,n':m_1,n_1;\cdots;m_r,n_r}_{p',q':p_1,q_1;\cdots;p_r,q_r}\left(\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix};(p,q)\left|\begin{matrix}\left(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j\right)_{1,p'}:\\ \\ \left(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j\right)_{1,q'}:\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\left(a_j^{(1)};\gamma_j^{(1)};A_j^{(1)}\right)_{1,p^{(1)}},\cdots,\left(a_j^{(r)};\gamma_j^{(r)};A_j^{(r)}\right)_{1,p_r}\\ \\ \left(b_j^{(1)};\delta_j^{(1)};B_j^{(1)}\right)_{1,q^{(1)}},\cdots,\left(b_j^{(r)};\delta_j^{(r)};B_j^{(r)}\right)_{1,q_r}\end{matrix}\right)$$
$$=\frac{1}{\left(2\pi\omega\right)^{r}}\int_{L_1}\cdots\int_{L_r}\psi\left(s_1,\cdots,s_r;p,q\right)\prod_{i=1}^{r}\phi_i\left(s_i;p,q\right)z_i^{s_i}\,\mathrm{d}s_1\cdots\mathrm{d}s_r,\tag{2.1}$$

where $\omega=\sqrt{-1}$,

$$\psi\left(s_1,\cdots,s_r;p,q\right)=\frac{\prod_{j=1}^{n'}\Gamma_{pq}^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n'+1}^{p'}\Gamma_{pq}^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q'}\Gamma_{pq}^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)},\tag{2.2}$$

$\phi(s_i;p,q)$

$$= \frac{\prod_{j=1}^{m_i}\Gamma_{pq}^{B_j^{(i)}}\left(b_j^{(i)}-\delta_j^{(i)}s_i\right)\prod_{j=1}^{n_i}\Gamma_{pq}^{A_j^{(i)}}\left(1-a_j^{(i)}+\gamma_j^{(i)}s_i\right)}{\prod_{j=1+m_i}^{q_i}\Gamma_{pq}^{B_j^{(i)}}\left(1-b_j^{(i)}+\delta_j^{(i)}s_i\right)\prod_{j=n_i+1}^{p_i}\Gamma_{pq}^{A_j^{(i)}}\left(a_j^{(i)}-\gamma_j^{(i)}s_i\right)\Gamma_{pq}(s_i)\Gamma_{pq}(1-s_i)\sin\pi s_i}. \tag{2.3}$$

An empty product is interpreted as unity; In the following. We replace n, p and q by n', p' and q' because the first three numbers are already used, see the Lemma 1.1.

The following numbers $A_j(j=1,\cdots,p')$, $B_j(j=1,\cdots,q')$, $A_j^{(i)}(i=1,\cdots,r;j=1,\cdots,p_i)$, $B_j^{(i)}(i=1,\cdots,r;j=1,\cdots,q_i)$, $\alpha_j^{(i)}(i=1,\cdots,r;j=1,\cdots,p')$, $\beta_j^{(i)}(i=1,\cdots,r;j=1,\cdots,q')$, $\gamma_j^{(i)}(i=1,\cdots,r;j=1,\cdots,p_i)$ and $\delta_j^{(i)}(i=1,\cdots,r;j=1,\cdots,q_i)$ are positive real numbers, the quantities $a_j(j=1,\cdots,p')$, $b_j(j=1,\cdots,q')$, $a_j^{(i)}(i=1,\cdots,r;j=1,\cdots,p_i)$ and $b_j^{(i)}(i=1,\cdots,r;j=1,\cdots,q_i)$ are complex numbers. The numbers m_i, n_i, p_i, q_i $(i=1,\cdots,r)$ are non-negative integers verifying the inequalities $0\leq q_i$, $0\leq m_i\leq q_i$, $0\leq n_i\leq p_i$ $(i=1,\cdots,r)$ and $0\leq n_i\leq p_i$.

Here (i) denotes the numbers of dashes. The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from to $s_k+i\sigma_k$ where σ_k if is a real number with loop, if necessary to ensure that the poles of $\Gamma_{pq}^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)$, $(j=1,\cdots,n')$, $\Gamma_{pq}^{A_j^{(k)}}(1-a_j^{(k)}+\gamma_j^{(k)}s_k)$, $(j=1,\cdots,n^{(k)})$, $(k=1,\cdots,r)$ to the left of the contour L_k and the poles of $\Gamma_{pq}^{B_j^{(k)}}(b_j^{(k)}-\delta_j^{(k)}s_k)$, $(j=1,\cdots,m^{(k)})$, $(k=1,\cdots,r)$ lie to the right of the contour L_k. The various parameters are restricted so that these poles of the integrand are assumed to be simple. The point $z_i=0(i=1,\cdots,r)$ is tacitly excluded. For large values of $|z_i|$, $Re(s_i log(z_i)-log\sin\pi s_i)<0$, $i=1,\cdots,r$.

We shall note

$$X=m_1,n_1;\cdots;m_r,n_r;\quad V=p_1,q_1;\cdots;p_r,q_r; \tag{2.4}$$

$$A=\left(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j\right)_{1,p'};\quad B=\left(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j\right)_{1,q'}; \tag{2.5}$$

$$C=\left(c_j^{(1)};\gamma_j^{(1)};C_j^{(1)}\right)_{1,p_1};\cdots;\left(c_j^{(r)};\gamma_j^{(r)};C_j^{(r)}\right)_{1,p_r}; \tag{2.6}$$

$$D=\left(d_j^{(1)};\delta_j^{(1)};D_j^{(1)}\right)_{1,q_1};\cdots;\left(d_j^{(r)};\delta_j^{(r)};D_j^{(r)}\right)_{1,q_r}. \tag{2.7}$$

3. Main formulae

In this section, we will establish one fractional (p,q)-derivative formula about the (p,q)-analogue of multivariable I-function defined by Prathima et al. [26] and studied by [13, 31].

Theorem 1. *Let $\Re(\mu) > 0, z_i \neq 0, \lambda_i > 0\,(i = 1, \cdots, r)$ and $0 < q < p$, then the following result holds true*

$$D^{\mu}_{p,q}\left\{ I\left(\begin{matrix} z_1 p x^{\lambda_1} \\ \vdots \\ z_r p x^{\lambda_r} \end{matrix} ;\, (p,q) \left| \begin{matrix} A : C \\ \\ B : D \end{matrix} \right. \right) \right\}$$
$$= x^{-\mu}\, I^{0,n'+1:X}_{p'+1,q'+1:V} \left(\begin{matrix} z_1 p x^{\lambda_1} \\ \vdots \\ z_r p x^{\lambda_r} \end{matrix} ;\, (p,q) \left| \begin{matrix} (0,\lambda_1,\cdots,\lambda_r;1), A : C \\ \\ B, (\mu;\lambda_1,\cdots,\lambda_r;1) : D \end{matrix} \right. \right), \tag{3.1}$$

where $\Re\left(t_i \log\left(z_i\right) - \log\sin \pi t_i\right) < 0 \;\; (i = 1, \cdots, r)$.

Proof. To prove (3.1), we consider the left-hand side of equation (3.1) (say L.H.S.) and make use of the formula (2.1), we obtain

$$\text{L.H.S.}$$
$$= D^{\mu}_{p,q}\left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \pi^r \psi\left(s_1, \cdots, s_r; p, q\right) \prod_{i=1}^{r} \phi_i\left(s_i; p, q\right) \left(z_i p x^{\lambda_i}\right)^{s_i} \mathrm{d}s_1 \cdots \mathrm{d}s_r \right\}. \tag{3.2}$$

Now, we interchange the order of $\int$ and $D^{\mu}_{p,q}$ because the validities conditions are satisfied. This gives

$$\text{L.H.S.}$$
$$= \int_{L_1} \cdots \int_{L_r} \pi^r \psi\left(s_1, \cdots, s_r; p, q\right) \prod_{i=1}^{r} \phi_i\left(s_i; p, q\right) D^{\mu}_{p,q}\left(\prod_{i=1}^{r} p^{s_i} z_i^{s_i} x^{\sum_{i=1}^{r} \lambda_i s_i} \right) \mathrm{d}s_1 \cdots \mathrm{d}s_r. \tag{3.3}$$

We can also write

$$\text{L.H.S.} =$$
$$\int_{L_1} \cdots \int_{L_r} \pi^r \psi\left(s_1, \cdots, s_r; p, q\right) \prod_{i=1}^{r} \phi_i\left(s_i; p, q\right) \left(p z_i\right)^{s_i} D^{\mu}_{p,q}\left(x^{\sum_{i=1}^{r} \lambda_i s_i} \right) \mathrm{d}s_1 \cdots \mathrm{d}s_r. \tag{3.4}$$

We use the Lemma 1.1 and after a few algebraic transformations, we get

$$\text{L.H.S.} = \int_{L_1} \cdots \int_{L_r} \pi^r \psi\left(s_1, \cdots, s_r; p, q\right) \prod_{i=1}^{r} \phi_i\left(s_i; p, q\right) \left(p z_i\right)^{s_i}$$
$$\times \frac{\Gamma_{p,q}\left(1 - 0 + \sum_{i=1}^{r} \lambda_i s_i\right)}{\Gamma_{p,q}\left(1 - \mu + \sum_{i=1}^{r} \lambda_i s_i\right)} x^{-\mu + \sum_{i=1}^{r} \lambda_i s_i} \mathrm{d}s_1 \cdots \mathrm{d}s_r. \tag{3.5}$$

Interpreting the above multiple Mellin-Barnes contour integrals in terms of the basic (p, q)-multivariable I-function, we get the desired result (3.1).

4. Special cases

In this section, we give several particular cases.

First, we suppose $p = 1$, then the (p, q)-basic multivariable I-function is replaced by the q-basic multivariable Prathima's I-function and we have the following relation:

Definition 4.1. *If* $\Re\left(\mu\right) > 0, z_i \neq 0$, $\lambda_i > 0$ $(i = 1, \cdots, r)$, $\Re\left(s_i \log\left(z_i\right) - \log\sin \pi s_i\right) < 0$, $(i = 1, \cdots, r)$ *and* $p = 1$. *Then q-basic multivariable Prathima's I-function defined by the multiple q-contour integrals, given by*

$$I\left(z_1, \cdots, z_r; q\right) = I^{0,n':m_1,n_1;\cdots;m_r,n_r}_{p',q':p_1,q_1;\cdots;p_r,q_r} \left(\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} ; q \,\middle|\, \begin{matrix} \left(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}; A_j\right)_{1,p'} : \\ \\ \left(b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}; B_j\right)_{1,q'} : \end{matrix} \right.$$
$$\left. \begin{matrix} \left(a_j^{(1)}; \alpha_j^{(1)}; A_j^{(1)}\right)_{1,p^{(1)}}, \cdots, \left(a_j^{(r)}; \alpha_j^{(r)}; A_j^{(r)}\right)_{1,p^{(r)}} \\ \\ \left(b_j^{(1)}; \beta_j^{(1)}; B_j^{(1)}\right)_{1,q^{(1)}}, \cdots, \left(b_j^{(r)}; \beta_j^{(r)}; B_j^{(r)}\right)_{1,q^{(r)}} \end{matrix} \right)$$
$$= \frac{1}{\left(2\pi\omega\right)^r} \int_{L_1} \cdots \int_{L_r} \psi\left(s_1, \cdots, s_r; q\right) \prod_{i=1}^{r} \phi_i\left(s_i; q\right) z_i^{s_i} \mathrm{d}s_1 \cdots \mathrm{d}s_r, \tag{4.1}$$

where

$$\psi\left(s_1, \cdots, s_r; q\right) = \frac{\prod_{j=1}^{n'} \Gamma_q^{A_j}\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} t_j\right)}{\prod_{j=n'+1}^{p'} \Gamma_q^{A_j}\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} t_j\right) \prod_{j=1}^{q'} \Gamma_q^{B_j}\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} t_j\right)}, \tag{4.2}$$

and

$$\phi(s_i;q) = \frac{\prod_{j=1}^{m^{(i)}}\Gamma_q^{B_j^{(i)}}\left(b_j^{(i)}-\beta_j^{(i)}s\right)\prod_{j=1}^{n^{(i)}}\Gamma_q^{A_j^{(i)}}\left(1-a_j^{(i)}+\alpha_j^{(i)}s_i\right)}{\prod_{j=1+m^{(i)}}^{q^{(i)}}\Gamma_q^{B_j^{(i)}}\left(1-b_j^{(i)}+\beta_j^{(i)}s_i\right)\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma_q^{A_j^{(i)}}\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)\Gamma_q(s_i)\Gamma_q(1-s_i)\sin\pi s_i}. \tag{4.3}$$

Corollary 1. *By using Definition 4.1 and the notations* (2.4)-(2.7) *with* $p = 1$, *then we obtain the following result:*

$$D_q^{\mu}\left\{ I\left(\begin{array}{c} z_1x^{\lambda_1} \\ \vdots \\ z_rx^{\lambda_r}\end{array};q \left| \begin{array}{c} A:C \\ \\ B:D\end{array}\right.\right)\right\} = x^{-\mu} I_{p'+1,q'+1:V}^{0,n'+1:X}\left(\begin{array}{c} z_1x^{\lambda_1} \\ \vdots \\ z_rx^{\lambda_r}\end{array};q \left| \begin{array}{c} (0,\lambda_1,\cdots,\lambda_r;1), A:C \\ \\ B,(\mu;\lambda_1,\cdots,\lambda_r;1):D\end{array}\right.\right). \tag{4.4}$$

Definition 4.2. *If* $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$, *then the* (p, q)*-basic multivariable* I*-function reduces to* (p, q)*-basic multivariable* H*-function defined by (for details of multivariable* H*-function, the readers can refer the work* [8, 9, 20, 33, 34]*)* Let

$$A_0 = \left(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)}\right)_{1,p'};\quad B_0 = \left(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)}\right)_{1,q'}; \tag{4.5}$$

$$C_0 = \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\cdots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r};\quad D_0 = \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\cdots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}. \tag{4.6}$$

Then we have the following result:

Corollary 2.

$$D_{p,q}^{\mu}\left\{ H\left(\begin{array}{c} z_1px^{\lambda_1} \\ \vdots \\ z_rpx^{\lambda_r}\end{array};(p,q) \left| \begin{array}{c} A_0:C_0 \\ \\ B_0:D_0\end{array}\right.\right)\right\} = x^{-\mu} H_{p'+1,q'+1:V}^{0,n'+1:X}\left(\begin{array}{c} z_1px^{\lambda_1} \\ \\ z_rpx^{\lambda_r}\end{array};(p,q) \left| \begin{array}{c} (0;\lambda_1,\cdots,\lambda_r), A_0:C_0 \\ \\ B_0,(\mu;\lambda_1,\cdots,\lambda_r):D_0\end{array}\right.\right), \tag{4.7}$$

under the same conditions stated in the Theorem 1 and the Definition 4.2.

Definition 4.3. *Taking $r = 2$, the (p, q)-basic multivariable I-function defined by (2.1) is replaced by the (p, q)-basic I-function of two variables defined by Kumari et al. [21]. This new function is defined in the following manner:*

$$I\left(z_1, z_2; p, q\right)$$
$$= I^{0,n_1,m_2,n_2,m_3,n_3}_{p_1,q_1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1 \\ \\ z_2 \end{matrix} ; (p,q) \left| \begin{matrix} (a_i;\alpha_i,A_i;\mathbf{A}_i)_{1,p_1} : (e_i;E_i;\mathbf{E}_i)_{1,p_2} ; (g_i;G_i;\mathbf{G}_i)_{1,p_3} \\ \\ (b_i;\beta_i,B_i;\mathbf{B}_i)_{1,q_1} : (f_i;F_i;\mathbf{F}_i)_{1,q_2} \, (h_i;H_i;\mathbf{H}_i)_{1,q_3} \end{matrix} \right.\right)$$
$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \psi\left(s,t;p,q\right)\, \phi_1\left(s;p,q\right)\, \phi_2\left(t;p,q\right)\, z_1^s z_2^t \, \mathrm{d}s\mathrm{d}t, \tag{4.8}$$

where

$$\psi\left(s,t;p,q\right) = \frac{\prod_{i=1}^{n_1}\Gamma_{pq}^{\mathbf{A}_i}\left(1-a_i+\alpha_i s + A_i t\right)}{\prod_{i=1}^{q_1}\Gamma_{pq}^{\mathbf{B}_i}\left(1-b_i+\beta_i s+B_j t\right)\prod_{i=n_1+1}^{p_1}\Gamma_{pq}^{\mathbf{A}_i}\left(a_i-\alpha_i s - A_i t\right)}, \tag{4.9}$$

$$\phi_1\left(s;p,q\right) = \frac{\prod_{i=1}^{m_2}\Gamma_{pq}^{\mathbf{F}_i}\left(f_i - F_i s\right)\prod_{i=1}^{n_2}\Gamma_{pq}^{\mathbf{E}_i}\left(1-e_i+E_i s\right)}{\prod_{i=m_2+1}^{q_2}\Gamma_{pq}^{\mathbf{F}_i}\left(1-f_i+F_i s\right)\prod_{i=n_2+1}^{p_2}\Gamma_{pq}^{\mathbf{E}_i}\left(e_i-E_i s\right)\Gamma_{pq}\left(s\right)\Gamma_{pq}\left(1-s\right)\sin ps}, \tag{4.10}$$

$$\phi_2\left(t;p,q\right) = \frac{\prod_{i=1}^{m_3}\Gamma_{pq}^{\mathbf{H}_i}\left(h_i - H_i t\right)\prod_{i=1}^{n_3}\Gamma_{pq}^{\mathbf{G}_i}\left(1-g_i+G_i t\right)}{\prod_{i=m_3+1}^{q_3}\Gamma_{pq}^{\mathbf{H}_i}\left(1-h_i+H_i t\right)\prod_{i=n_3+1}^{p_3}\Gamma_{pq}^{\mathbf{G}_i}\left(g_i-G_i t\right)\Gamma_{pq}\left(t\right)\Gamma_{pq}\left(1-t\right)\sin \pi t}. \tag{4.11}$$

We use the following notations:

$$A_1 = (a_i;\alpha_i,A_i;\mathbf{A}_i)_{1,p_1}\,; \quad A_2 = (e_i;E_i;\mathbf{E}_i)_{1,p_2}\,;(g_i;G_i;\mathbf{G}_i)_{1,p_3}\,; \tag{4.12}$$

$$B_1 = (b_i;\beta_i,B_i;\mathbf{B}_i)_{1,q_1}\,; \quad B_2 = (f_i;F_i;\mathbf{F}_i)_{1,q_2}\,;(h_i;H_i;\mathbf{H}_i)_{1,q_3}\,. \tag{4.13}$$

Then, we have the result as follows:

Corollary 3.

$$D^{\mu}_{p,q}\left\{ I^{0,n_1,m_2,n_2,m_3,n_3}_{p_1,q_1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1 p x^{\lambda_1} \\ \\ z_2 p x^{\lambda_2}\end{matrix} ; (p,q) \left| \begin{matrix} A_1 ; A_2 \\ \\ B_1 ; B_2 \end{matrix}\right.\right)\right\}$$
$$= x^{-\mu}\, I^{0,n_1+1,m_2,n_2,m_3,n_3}_{p_1+1,q_1+1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1 p x^{\lambda_1} \\ \\ z_2 p x^{\lambda_2}\end{matrix} ; (p,q) \left| \begin{matrix} \left(0;\lambda_1,\lambda_2;1\right), A_1 : A_2 \\ \\ B_1, \left(\mu;\lambda_1,\lambda_2;1\right); B_2 \end{matrix}\right.\right), \tag{4.14}$$

where $\Re\left(\mu\right) > 0,\ \lambda_i > 0,\ \Re\left(t_i \log\left(z_i\right) - \log\sin\pi t_i\right) < 0 \quad (i = 1, 2)$.
Next, if we set $\mathbf{A_i} = \mathbf{B_i} = \mathbf{E_i} = \mathbf{F_i} = \mathbf{G_i} = \mathbf{H_i} = 1$ in (4.8) then (p, q)-basic I-function of two variables reduces to (p, q)- analogue of H-function of two variables

[27].
Let

$$\mathbf{A}_1 = (a_i;\alpha_i,A_i)_{1,p_1}\,;\quad \mathbf{A}_2 = (e_i;E_i)_{1,p_2}\,;(g_i;G_i)_{1,p_3}\,; \tag{4.15}$$

$$\mathbf{B}_1 = (b_i;\beta_i,B_i)_{1,q_1}\,;\quad \mathbf{B}_2 = (f_i;F_i)_{1,q_2}\,;(h_i;H_i)_{1,q_3}\,. \tag{4.16}$$

We obtain the following formula:

Corollary 4.

$$D^{\mu}_{p,q}\left\{H^{0,n_1,m_2,n_2,m_3,n_3}_{p_1,q_1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1px_1^{\lambda_1} \\ \\ z_2px_2^{\lambda_2}\end{matrix}\;;(p,q)\,\middle|\,\begin{matrix}\mathbf{A}_1:\mathbf{A}_2 \\ \\ \mathbf{B}_1:\mathbf{B}_2\end{matrix}\right)\right\}$$
$$= x^{-\mu}\,H^{0,n_1+1,m_2,n_2,m_3,n_3}_{p_1+1,q_1+1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1px_1^{\lambda_1} \\ \\ z_2px_2^{\lambda_2}\end{matrix}\;;(p,q)\,\middle|\,\begin{matrix}(0;\lambda_1,\lambda_2)\,,\mathbf{A}_1:\mathbf{A}_2 \\ \\ \mathbf{B}_1,(\mu;\lambda_1,\lambda_2):\mathbf{B}_2\end{matrix}\right). \tag{4.17}$$

valid under the conditions stated in Corollary 3.
If we take $(\alpha_i)_{1,p_1} = (A_i)_{1,p_1} = (E_i)_{1,p_2} = (G_i)_{1,p_3} = (\beta_i)_{1,q_1} = (B_i)_{1,q_1} = (F_i)_{1,q_2} = (H)_{1,q_3} = 1$, then the (p,q)- analogue of H-function of two variables reduces to (p,q)- analogue of Meijer's G-function of two variables [1].
Let

$$\mathbf{A}_0 = (a_i)_{1,p_1}\,;\quad \mathbf{A}'_0 = (e_i)_{1,p_2}\,;(g)_{1,p_3}\,;\quad \mathbf{B}_0 = (b_i)_{1,q_1}\,;\quad \mathbf{B}'_0 = (f_i)_{1,q_2}\,;(h_i)_{1,q_3}\,, \tag{4.18}$$

then we obtain the following relation:

Corollary 5.

$$D^{\mu}_{p,q}\left\{G^{0,n_1,m_2,n_2,m_3,n_3}_{p_1,q_1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1px^{\lambda_1} \\ \\ z_2px^{\lambda_2}\end{matrix}\;;(p,q)\,\middle|\,\begin{matrix}\mathbf{A}_0;\mathbf{A}'_0 \\ \\ \mathbf{B}_0;\mathbf{B}'_0\end{matrix}\right)\right\}$$
$$= x^{-\mu}\,G^{0,n_1+1,m_2,n_2,m_3,n_3}_{p_1+1,q_1+1,p_2,q_2;p_3,q_3}\left(\begin{matrix} z_1px_1^{\lambda_1} \\ \\ z_2px_2^{\lambda_2}\end{matrix}\;;(p,q)\,\middle|\,\begin{matrix}(0;\lambda_1,\lambda_2)\,,\mathbf{A}_0:\mathbf{A}'_0 \\ \\ \mathbf{B}_0,(\mu;\lambda_1,\lambda_2):\mathbf{B}'_0\end{matrix}\right). \tag{4.19}$$

If $r=1$, then (p,q)-basic multivariable I-function reduces to (p,q)- basic I-function of one variable defined by Rathie [28].
We have the following result:

Corollary 6.

$$D^{\mu}_{p,q}\left\{I^{m',n'}_{p',q'}\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(a_j,\alpha_j:A_j)_{1,p'}\\ \\ (b_j,\beta_j:B_j)_{1,q'}\end{matrix}\right)\right\}$$

$$=x^{-\mu}\,I^{m',n'+1}_{p'+1,q'+1}\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(0;\rho;1)\,,(a_j,\alpha_j:A_j)_{1,p'}\\ \\ (b_j,\beta_j:B_j)_{1,q'}\,,(\mu;\rho;1)\end{matrix}\right), \tag{4.20}$$

where $\Re\left(s\log\left(z\right)-\log\left(\pi s\right)\right)<0$, *the conditions of Theorem 1 are verified and* $r=1$.

If we take $r=1$ and $A_j=B_j=1$, then the (p,q)- basic analogue of I-function of one variable reduces to (p,q)- basic analogue of H-function of one variable [10], we have

Corollary 7.

$$D^{\mu}_{p,q}\left\{H^{m',n'}_{p',q'}\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,p'}\\ \\ (b_j,\beta_j)_{1,q'}\end{matrix}\right)\right\}$$

$$=x^{-\mu}\,H^{m'+1,n'+1}_{p'+1,q'+1}\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(0;\rho)\,,(a_j,\alpha_j)_{1,p'}\\ \\ (b_j,\beta_j)_{1,q'}\,,(\mu;\rho)\end{matrix}\right), \tag{4.21}$$

where $\rho>0$ *and* $\Re\left(s\log\left(z\right)-\log\sin\pi s\right)<0$.

Now, we suppose $(\alpha_j)_{1,p'}=(\beta_j)_{1,q'}=1$, then the (p,q)- basic analogue of H-function reduces to (p,q)- basic analogue of Meijer's G-function [23]. Then we have the following result:

Corollary 8.

$$D^{\mu}_{p,q}\left\{G^{m',n'}_{p',q'}\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(a_j)_{1,p'}\\ \\ (b_j)_{1,q'}\end{matrix}\right)\right\}$$

$$=x^{-\mu}\,G^{m',n'+1}_{p'+1,q'+1}\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(0;\rho)\,,(a_j)_{1,p'}\\ \\ (b_j)_{1,q'}\,,(\mu;\rho)\end{matrix}\right), \tag{4.22}$$

under the conditions verified by the Corollary 7 and $(\alpha_j)_{1,p'}=(\beta_j)_{1,q'}=1$.

Definition 4.4. *We have a relation concerning the (p, q)- basic analogue of Meijer's G-function and (p, q)- basic analogue of Mac-Robert's E-function* [22], *given by*

$$G_{q'+1,p'}^{p'+1,1}\left(z;\,(p,q)\,\middle|\,\begin{matrix}1,(a_j)_{1,p'}\\(b_j)_{1,q'}\end{matrix}\right)=E\left(z;\,(p,q)\,\middle|\,\begin{matrix}(b_j)_{1,q'}\\(a_j)_{1,p'}\end{matrix}\right). \tag{4.23}$$

Consequently, we obtain the following formula about the Mac-Robert's E- function:

Corollary 9. *Let $\Re(\mu) > 0$ and $z \neq 0$, then the following result holds true*

$$D_{p,q}^{\mu}\left\{E\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(a_j)_{1,p'}\\(b_j)_{1,q'}\end{matrix}\right)\right\}=x^{-\mu}\,E\left(zpx^{\rho};\,(p,q)\,\middle|\,\begin{matrix}(0;\rho),(a_j)_{1,p'}\\(\mu;\rho)\end{matrix}\right), \tag{4.24}$$

where $\Re(t\log(z)-\log\sin\pi t)<0$.

Remark 4.1. *We have the identical formulas with the (p, q)- basic analogue of multivariable Gimel-function, see* [6], *the (p, q)-basic analogue of multivariable I-function* [12, 25, 31], *the (p, q)- basic analogue of the modified multivariable H-function* [8, 9, 17, 20], *the (p, q)- basic analogue of the multivariable Aleph-function* [15], *the (p, q)- basic analogue of the multivariable A-function* [11, 14]. *By using the same method, we can obtain the equivalent results concerning the others (p, q)-basic analogue of the special functions of one and several variables, see* [4] *for more details.*

5. Conclusion

In this paper, the (p, q)- basic analogue of multivariable I-function with (p, q) derivative has been derived by using the Gamma and Beta functions. These results are very useful as they are most general in the nature. By choosing some particular values of the various parameters as well as variables in the (p, q)- basic analogue of multivariable I-function, we get a big number of results having remarkably wide variety of useful (p, q)- basic analogue of special functions (or product of such (p, q)-basic analogue functions) which can be expressed in terms of (p, q)- basic analogue of I-function, (p, q)- basic analogue of H-function, (p, q)- basic analogue of Meijer's G-function, (p, q)- basic analogue of E-function and (p, q)- basic analogue of special functions of one and several variables. The (p, q)-calculus is frequently used in quantum physics, quantum calculus [7, 24] and several domains about the mathematics.

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